

# CONVEX POLYNOMIAL APPROXIMATION IN $\mathbb{R}^d$ WITH FREUD WEIGHTS

O. MAIZLISH AND A. PRYMAK

**ABSTRACT.** We show that for multivariate Freud-type weights  $W_\alpha(\mathbf{x}) = \exp(-|\mathbf{x}|^\alpha)$ ,  $\alpha > 1$ , any convex function  $f$  on  $\mathbb{R}^d$  satisfying  $fW_\alpha \in L_p(\mathbb{R}^d)$  if  $1 \leq p < \infty$ , or  $\lim_{|\mathbf{x}| \rightarrow \infty} f(\mathbf{x})W_\alpha(\mathbf{x}) = 0$  if  $p = \infty$ , can be approximated in the weighted norm by a sequence  $P_n$  of algebraic polynomials convex on  $\mathbb{R}^d$  such that  $\|(f - P_n)W_\alpha\|_{L_p(\mathbb{R}^d)} \rightarrow 0$  as  $n \rightarrow \infty$ . This extends the previously known result for  $d = 1$  and  $p = \infty$  obtained by the first author to higher dimensions and integral norms using a completely different approach.

*Keywords:* Multivariate weighted approximation; Freud weights; convex approximation

## 1. INTRODUCTION

In this paper, we study multivariate polynomial approximation with exponential weights. Given a continuous weight function  $W : \mathbb{R}^d \rightarrow (0, 1]$ ,  $d \geq 1$ , consider the class of real-valued functions  $C_W(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \lim_{|\mathbf{x}| \rightarrow \infty} f(\mathbf{x})W(\mathbf{x}) = 0\}$ , where  $|\mathbf{x}| = |(x_1, \dots, x_d)| := (x_1^2 + x_2^2 + \dots + x_d^2)^{1/2}$ . The density of algebraic polynomials in the space  $C_W(\mathbb{R}^d)$  has been established by Kroo in [4] for various exponential weights  $W(\mathbf{x}) = e^{-Q(\mathbf{x})}$ . It was shown that under some technical assumptions on the function  $Q(\mathbf{x})$  derived from the univariate case, the set of algebraic polynomials in  $d$  variables is dense in  $C_W(\mathbb{R}^d)$  equipped with the weighted norm  $\|f\|_W := \sup_{\mathbf{x} \in \mathbb{R}^d} |f(\mathbf{x})W(\mathbf{x})|$  if and only if for every  $i$ ,  $1 \leq i \leq d$ ,

$$\int_{-\infty}^{\infty} \frac{Q(0, \dots, 0, x_i, 0, \dots, 0)}{1 + x_i^2} dx_i = \infty.$$

In particular, the so-called (multivariate) Freud weights  $W_\alpha(\mathbf{x}) := e^{-|\mathbf{x}|^\alpha}$  satisfy the above condition for  $\alpha \geq 1$ . Unlike in the multivariate case, the univariate theory of approximation with exponential weights is well studied and fairly complete, see, for instance, survey [7] by Lubinsky.

A typical problem of shape preserving approximation asks whether one can approximate a function possessing a certain geometric property (e.g., monotonicity or convexity) by a polynomial having the same property. A survey on shape preserving approximation of functions of one variable is given in [3].

---

Both authors were supported in part by NSERC of Canada. The postdoctoral fellowship of the first author was also partially funded by the Department of Mathematics of the University of Manitoba.

Two recent articles studied possibility of shape preserving approximation with Freud weights on the real line: Maizlish [8] showed density for the so-called  $k$ -monotone polynomial approximation in the space  $C_{W_\alpha}(\mathbb{R})$ , then Leviatan and Lubinsky [6] established Jackson-type estimates on this kind of approximation. (On the real line, 1-monotone and 2-monotone functions are the usual monotone and convex functions, respectively.) In the present paper, we obtain a multivariate analogue of the result from [8] for convex approximation and extend the treatment to the integral norms.

Let  $S \subset \mathbb{R}^d$  be a measurable set. As usual, denote by  $L_p(S)$ ,  $1 \leq p \leq \infty$ , the space of all measurable functions  $f : S \rightarrow \mathbb{R}$  such that  $\|f\|_{L_p(S)} < \infty$ , where  $\|f\|_{L_p(S)} := (\int_S |f(\mathbf{x})|^p d\mathbf{x})^{1/p}$  if  $1 \leq p < \infty$ , and  $\|f\|_{L_\infty(S)} := \text{ess sup}_{\mathbf{x} \in S} |f(\mathbf{x})|$  if  $p = \infty$ . If  $S \subset \mathbb{R}^d$  is convex (i.e., for any  $\mathbf{x}, \mathbf{y} \in S$  the straight line segment joining  $\mathbf{x}$  and  $\mathbf{y}$  is also in  $S$ ), we recall that a function  $f : S \rightarrow \mathbb{R}$  is called convex on  $S$  if  $f(\theta\mathbf{x} + (1 - \theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1 - \theta)f(\mathbf{y})$  for any  $\mathbf{x}, \mathbf{y} \in S$ ,  $\theta \in [0, 1]$ . Alternative criteria for convexity of twice continuously differentiable functions on an open convex domain are: positive semi-definiteness of the Hessian or non-negativity of any second directional derivative. If  $f : S \rightarrow \mathbb{R}$  is convex on  $S$ , then  $f$  is continuous on the interior of  $S$ . Denote by  $\Pi_{n,d}$  the space of algebraic polynomials of total degree  $\leq n$  in  $d$  variables. Let  $\Pi_{n,d}^{(2)}$  be the set of all polynomials from  $\Pi_{n,d}$  that are convex on  $\mathbb{R}^d$ .

Now we are ready to state the main result.

**Theorem 1.** *Let  $1 \leq p \leq \infty$  and  $\alpha > 1$ . Suppose that  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is convex on  $\mathbb{R}^d$ , and, in addition,  $fW_\alpha \in L_p(\mathbb{R}^d)$  if  $1 \leq p < \infty$ , or  $f \in C_{W_\alpha}(\mathbb{R}^d)$  if  $p = \infty$ . Then*

$$(1) \quad \inf_{P \in \Pi_{n,d}^{(2)}} \|(f - P)W_\alpha\|_{L_p(\mathbb{R}^d)} \rightarrow 0, \quad n \rightarrow \infty.$$

We remark that for any polynomial  $P \in \Pi_{n,d}$  and  $\alpha \geq 1$ , we have  $PW_\alpha \in L_p(\mathbb{R}^d)$  and  $P \in C_{W_\alpha}(\mathbb{R}^d)$ , therefore the restrictions on  $f$  in the above theorem are a natural assumption to consider approximation in the weighted norm.

Note that in [4] a similar result to (1) for *unconstrained* approximation was obtained for  $p = \infty$  only (but for a wider class of weights). For  $1 \leq p < \infty$ , density of algebraic polynomials in the weighted  $L_p$ -norm also follows from [4]. For instance, if  $fW_\alpha \in L_p(\mathbb{R}^d)$ ,  $\alpha > 1$ , we can approximate  $fW_\alpha$  in  $L_p(\mathbb{R}^d)$  by a function  $g \in C(\mathbb{R}^d)$  with bounded support, and then apply the result of [4] for  $gW_\alpha^{-1}$  to find an approximating polynomial  $P$  in  $C_{W_\beta}(\mathbb{R}^d)$ , where  $1 \leq \beta < \alpha$ . Then density follows from the estimate

$$\|(f - P)W_\alpha\|_{L_p(\mathbb{R}^d)} \leq \|fW_\alpha - g\|_{L_p(\mathbb{R}^d)} + \|(gW_\alpha^{-1} - P)W_\beta\|_{L_\infty(\mathbb{R}^d)} \|W_\alpha W_\beta^{-1}\|_{L_p(\mathbb{R}^d)},$$

where  $\|W_\alpha W_\beta^{-1}\|_{L_p(\mathbb{R}^d)} < \infty$ . Such reduction to the case  $p = \infty$  will not transfer to convexity preserving approximation since continuous functions of bounded support are not convex in general.

A very interesting open question is whether Theorem 1 remains valid for  $\alpha = 1$ , i.e., for the weight  $W_1(\mathbf{x}) = e^{-|\mathbf{x}|}$ ? The answer is not known even for  $d = 1$  and  $p = \infty$ . It is not clear either if similar results to [6] and [8] can be established for the weight  $W_1$ . It is worth noting that  $W_1$  is in a certain sense “boundary” case of the Freud weights, and many phenomena behave differently when  $\alpha = 1$ , see [7, Section 5]. Nevertheless, algebraic polynomials are dense in  $C_{W_1}(\mathbb{R}^d)$ .

A Jackson-type estimate for non-weighted multivariate convex polynomial approximation was established by Shvedov [10] in 1981, and no significant improvement of this estimate has been obtained to date. The main result of [10] as well as some ideas of the proofs were extremely important for this work.

Our proof of Theorem 1 is constructive, i.e., it contains a specific procedure for construction of the approximating convex polynomial. While it may be possible to derive an estimate on the error of approximation using our construction, such an estimate would be impractical, therefore we restricted ourselves only to establishing density in this paper. It will likely require a significantly different approach to obtain a reasonable (e.g. Jackson-type) quantitative estimate.

In Section 2, we state some known results and prove auxiliary lemmas concerning piecewise linear convex weighted approximation and multivariate versions of restricted range inequalities. In Section 3, we first outline the proof of Theorem 1 highlighting the main ideas without technicalities, then give all the details of the proof.

## 2. AUXILIARY LEMMAS

We start with some notations and definitions. By  $B(r) := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \leq r\}$  we denote the closed ball of radius  $r > 0$  in  $\mathbb{R}^d$  centered at the origin. The modulus of continuity of  $f : M \rightarrow \mathbb{R}$  is defined as

$$\omega(f, t, M) := \sup_{\mathbf{x}', \mathbf{x}'' \in M : |\mathbf{x}' - \mathbf{x}''| < t} |f(\mathbf{x}') - f(\mathbf{x}'')|.$$

A function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  is called a *piecewise linear convex* function, if  $g$  is the pointwise maximum of *finitely* many linear functions (elements of  $\Pi_{1,d}$ ) on  $\mathbb{R}^d$ .

The following two lemmas are particular cases of the results from [10] applied on  $B(r)$  and stated in our notations.

**Lemma 2** ([10, Lemma 3], piecewise linear convex approximation). *Let  $f : B(r) \rightarrow \mathbb{R}$  be convex and  $r > 0$  be fixed. Then for any  $\delta \in (0, 1]$ , there exists a piecewise linear convex function  $g_\delta$  such that*

$$(2) \quad g_\delta(\mathbf{x}) \leq f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$

and

$$\|f - g_\delta\|_{L_\infty(B(r))} \leq c_1 \omega(f, r\delta, B(r)),$$

where  $c_1 > 0$  is a constant depending only on  $d$ .

**Lemma 3** ([10, Theorem 1], Jackson-type estimate on convex multivariate approximation). *Let  $f : B(r) \rightarrow \mathbb{R}$  be convex and  $r > 0$  be fixed. Then for any positive integer  $n$ , there exists a polynomial  $P_n \in \Pi_{n,d}$  convex on  $B(r)$  such that*

$$\|f - P_n\|_{L_\infty(B(r))} \leq c_2 \omega\left(f, \frac{r}{n+1}, B(r)\right),$$

where  $c_2 > 0$  is a constant depending only on  $d$ .

It is important to explain certain points regarding Lemmas 2 and 3. First, in [10] Shvedov defines the modulus of continuity in a slightly different way (using the Minkowski functional of the domain) which was accounted for in the statements of the above two lemmas. Secondly, the fact that  $g_\delta$  is a piecewise linear convex function and the property (2) are not explicitly stated in [10, Lemma 3], but readily follow from the proof of [10, Lemma 3].

Now we prove an analogue of Lemma 2 for weighted approximation on  $\mathbb{R}^d$ .

**Lemma 4** (Piecewise linear convex weighted approximation). *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be convex,  $1 \leq p \leq \infty$  and  $\alpha \geq 1$ . Assume that  $fW_\alpha \in L_p(\mathbb{R}^d)$  if  $1 \leq p < \infty$ , or  $f \in C_{W_\alpha}(\mathbb{R}^d)$  if  $p = \infty$ . Then, for any  $\varepsilon > 0$ , there exist a piecewise linear convex function  $h$  such that*

$$(3) \quad \|(f - h)W_\alpha\|_{L_p(\mathbb{R}^d)} < \varepsilon$$

and

$$(4) \quad \omega(h, t, \mathbb{R}^d) \leq Lt \quad \text{for any } t > 0,$$

where  $L > 0$  does not depend on  $t$ .

*Proof.* Denote by  $l$  a linear function with the graph that is a supporting hyperplane for the graph of  $y = f(\mathbf{x})$  at the origin (we choose one of possibly many such linear functions). Then  $l(\mathbf{0}) = f(\mathbf{0})$  and  $l(\mathbf{x}) \leq f(\mathbf{x})$ ,  $\mathbf{x} \in \mathbb{R}^d$ .

Both functions  $f$  and  $l$  are clearly in  $C_{W_\alpha}(\mathbb{R}^d)$  if  $p = \infty$  or have finite weighted  $L_p$ -norms if  $1 \leq p < \infty$ . Thus, we can choose a sufficiently large  $r > 0$  such that

$$\max\{\|fW_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r))}, \|lW_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r))}\} < \varepsilon/4.$$

By Lemma 2, there exists a piecewise linear convex function  $g_\delta$  ( $\delta$  to be specified in a moment) such that

$$\|f - g_\delta\|_{L_\infty(B(r))} \leq c_1 \omega(f, r\delta, B(r)).$$

As  $f$  is convex on  $\mathbb{R}^d$ , it is continuous everywhere. Therefore, since  $r$  is fixed and  $B(r)$  is a compact set, we have  $\omega(f, r\delta, B(r)) \rightarrow 0$  as  $\delta \rightarrow 0+$ . Hence, we can choose a sufficiently small  $\delta > 0$  such that

$$\|f - g_\delta\|_{L_\infty(B(r))} < \frac{\varepsilon}{2\|W_\alpha\|_{L_p(B(r))}}.$$

We now define  $h(\mathbf{x}) := \max\{g_\delta(\mathbf{x}), l(\mathbf{x})\}$ ,  $\mathbf{x} \in \mathbb{R}^d$ , which is clearly a piecewise linear convex function. This definition and (2) imply

$$g_\delta(\mathbf{x}) \leq h(\mathbf{x}) \leq f(\mathbf{x}) \quad \text{and} \quad l(\mathbf{x}) \leq h(\mathbf{x}) \leq f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.$$

Hence, we can conclude that

$$\|(f - h)W_\alpha\|_{L_p(B(r))} \leq \|f - h\|_{L_\infty(B(r))}\|W_\alpha\|_{L_p(B(r))} \leq \|f - g_\delta\|_{L_\infty(B(r))}\|W_\alpha\|_{L_p(B(r))} < \varepsilon/2,$$

and

$$\|(f - h)W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r))} \leq \|(f - l)W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r))} \leq \|fW_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r))} + \|lW_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r))} < \varepsilon/2,$$

implying (3).

As  $h$  is a piecewise linear convex function, it is the maximum of a *finite* number of linear functions, therefore (4) follows immediately.  $\square$

The bound (4) means that  $h$  satisfies a Lipschitz condition of order one on the whole  $\mathbb{R}^d$ . By choosing the ratio  $\frac{r}{n+1}$  to be “small”, (4) ensures that the error of polynomial approximation of  $h$  on  $B(r)$  provided by Lemma 3 can be also “small” even if the radius  $r$  is “large”.

Another important ingredient for the proof of the main result is restricted-range inequalities. On the real line, these inequalities establish the estimates on the weighted norm of any polynomial outside a fixed finite interval in terms of the weighted norm of this polynomial inside the interval. An overview on this subject in the univariate case can be found in [7, Section 6]. We now state the multivariate generalizations of these inequalities which, to the knowledge of the authors, were first studied by Ganzburg in [2].

Let us recall the notions of the Freud and Mhaskar-Rakhmanov-Saff numbers. These numbers are defined for some classes of exponential weights  $W(\mathbf{x}) = e^{-Q(\mathbf{x})}$ , we however focus only on the case of Freud weights  $W_\alpha(\mathbf{x})$ ,  $\alpha > 1$ . For  $n \geq 1$ ,  $q_n := (n/\alpha)^{1/\alpha}$  denotes the  $n$ th Freud number, and  $a_n := \left(2^{\alpha-2} \frac{\Gamma(\alpha/2)^2}{\Gamma(\alpha)}\right)^{1/\alpha} n^{1/\alpha}$  denotes the  $n$ th Mhaskar-Rakhmanov-Saff number. Ignoring the constant factors, both values have the asymptotic behavior of  $n^{1/\alpha}$  as  $n \rightarrow \infty$ .

**Lemma 5** (Multivariate restricted-range inequalities). *Let  $\alpha > 1$  be fixed. Then for any  $P_n \in \Pi_{n,d}$ ,*

$$(5) \quad \|P_n W_\alpha\|_{L_\infty(\mathbb{R}^d \setminus B(4q_{2n}))} \leq 2^{-n} \|P_n W_\alpha\|_{L_\infty(B(q_{2n}))}$$

and, for any  $p$ ,  $1 \leq p < \infty$ , there exist positive constants  $c_3$ ,  $c_4$  and  $\delta$  (depending only on  $\alpha$ ,  $p$ , and  $d$ ) such that

$$(6) \quad \|P_n W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(2a_n))} \leq c_3 e^{-c_4 n^\delta} \|P_n W_\alpha\|_{L_p(B(a_n))}.$$

*Proof.* The inequality (5) is a direct corollary of the corresponding univariate inequalities. Indeed, for any direction  $\boldsymbol{\mu} \in \mathbb{R}^d$ ,  $|\boldsymbol{\mu}| = 1$ , it is enough to consider a univariate polynomial  $\tilde{P}_n(t) := P_n(t\boldsymbol{\mu})$  in  $t$  and apply the last inequality from the proof of [7, Theorem 6.1].

For some positive constants  $c_3$ ,  $c_4$ ,  $\delta_1$ , and for any  $\delta_2 \in (0, 2/3)$ , the inequality

$$\|P_n W_\alpha\|_{L_p(\mathbb{R}^d \setminus B((1+n^{\delta_2-2/3})a_n))} \leq c_3 e^{-c_4 n^{\delta_1}} \|P_n W_\alpha\|_{L_p(B(a_n))}$$

was established as the last inequality in the proof of [2, Theorem 2], where  $\Omega := \mathbb{R}^d$ ,  $\Omega_n := B(a_n)$ . With  $\delta = \delta_1$  and arbitrary choice of  $\delta_2 \in (0, 2/3)$ , we have  $n^{\delta_2-2/3} \leq 1$ , and (6) follows.  $\square$

Next tool to be used in the proof of the main result is an estimate on the growth of the second directional derivatives of a polynomial.

**Lemma 6.** *Let  $P_n \in \Pi_{n,d}$ . Then for any  $r > 0$  and any direction  $\boldsymbol{\mu}$ ,  $|\boldsymbol{\mu}| = 1$ , we have*

$$(7) \quad \left| \frac{\partial^2 P_n(\mathbf{x})}{\partial \boldsymbol{\mu}^2} \right| \leq \left( \frac{8|\mathbf{x}|}{r} \right)^n \frac{8n(n-1)}{r^2} \|P_n\|_{L_\infty(B(r))}, \quad |\mathbf{x}| \geq r/4.$$

*Proof.* Let  $r > 0$  be fixed. Observe that for any direction  $\boldsymbol{\mu} \in \mathbb{R}^d$ ,  $|\boldsymbol{\mu}| = 1$ , and point  $\mathbf{x} \in \mathbb{R}^d$ ,  $|\mathbf{x}| \leq r/2$ , both points  $\mathbf{x} \pm \frac{r}{2}\boldsymbol{\mu}$  belong to  $B(r)$ . Therefore, for any  $P_n \in \Pi_{n,d}$ , we can apply the Bernstein inequality  $|\tilde{P}'_n(0)| \leq n \|\tilde{P}_n\|_{L_\infty([-1,1])}$  to the univariate polynomial  $\tilde{P}_n(t) := P_n(\mathbf{x} + \frac{tr}{2}\boldsymbol{\mu})$ ,  $t \in \mathbb{R}$ , and obtain

$$\left| \frac{\partial P_n(\mathbf{x})}{\partial \boldsymbol{\mu}} \right| \leq \frac{2n}{r} \|P_n\|_{L_\infty(B(r))}, \quad |\mathbf{x}| \leq r/2.$$

Applying this inequality again for  $\frac{\partial P_n(\mathbf{x})}{\partial \boldsymbol{\mu}}$  (which is a polynomial in  $\mathbf{x}$  of total degree  $< n$ ) and  $\mathbf{x}$ ,  $|\mathbf{x}| \leq r/4$ , we obtain

$$(8) \quad \left| \frac{\partial^2 P_n(\mathbf{x})}{\partial \boldsymbol{\mu}^2} \right| \leq \frac{4(n-1)}{r} \left\| \frac{\partial P_n}{\partial \boldsymbol{\mu}} \right\|_{L_\infty(B(r/2))} \leq \frac{8n(n-1)}{r^2} \|P_n\|_{L_\infty(B(r))}, \quad |\mathbf{x}| \leq r/4.$$

We now fix  $\mathbf{x}$  such that  $|\mathbf{x}| \geq r/4$ . The function  $R(t) := \frac{\partial^2 P_n}{\partial \boldsymbol{\mu}^2}(t\mathbf{x}/|\mathbf{x}|)$  is a polynomial of a single variable  $t$  of degree  $< n$ . Comparing this polynomial with the corresponding Chebyshev polynomial (for instance, see [1, (2.10), p. 101]), we get

$$|R(t)| \leq \left( \frac{2|t|}{r/4} \right)^n \|R\|_{L_\infty([-r/4, r/4])}, \quad |t| \geq r/4.$$

Substituting  $t = |\mathbf{x}|$  in the inequality above, we obtain

$$\left| \frac{\partial^2 P_n(\mathbf{x})}{\partial \boldsymbol{\mu}^2} \right| \leq \left( \frac{8|\mathbf{x}|}{r} \right)^n \|R\|_{L_\infty([-r/4, r/4])}, \quad |\mathbf{x}| \geq r/4.$$

Taking into account

$$|R(t)| = \left| \frac{\partial^2 P_n(t\mathbf{x}/|\mathbf{x}|)}{\partial \boldsymbol{\mu}^2} \right| \leq \left\| \frac{\partial^2 P_n}{\partial \boldsymbol{\mu}^2} \right\|_{L_\infty(B(r/4))}, \quad |t| \leq r/4,$$

and (8), we have

$$\left| \frac{\partial^2 P_n(\mathbf{x})}{\partial \boldsymbol{\mu}^2} \right| \leq \left( \frac{8|\mathbf{x}|}{r} \right)^n \left\| \frac{\partial^2 P_n}{\partial \boldsymbol{\mu}^2} \right\|_{L_\infty(B(r/4))} \leq \left( \frac{8|\mathbf{x}|}{r} \right)^n \frac{8n(n-1)}{r^2} \|P_n\|_{L_\infty(B(r))}, \quad |\mathbf{x}| \geq r/4,$$

and the proof of the lemma is complete.  $\square$

It is also possible to obtain (8) from the results of [5] or [9], but the direct proof is short, so we included it here for completeness.

### 3. PROOF OF THEOREM 1

We begin with a sketch of the proof of Theorem 1. Given convex  $f$  in the proper weighted class, we apply Lemma 4 and obtain a piecewise linear convex  $h$  “close” to  $f$  in the weighted norm. The next step is to apply Lemma 3 to the function  $h$  with  $r := r_n$  to be chosen later. Since  $h$  satisfies (4), if  $\frac{r_n}{n+1} \rightarrow 0$  as  $n \rightarrow \infty$ , we find a sequence of polynomials  $P_n$  convex on  $B(r_n)$  that approximate  $h$  uniformly on  $B(r_n)$  with certain rate. This fact together with the estimates on the weighted norm of  $P_n$  outside  $B(r_n)$  (provided by the multivariate restricted-range inequalities given in Lemma 5) imply that  $P_n$  is “close” to  $h$  in the weighted norm. The goal is to modify  $P_n$  in such a way that the resulting polynomial is convex on the whole  $\mathbb{R}^d$ . Using the idea of the proof of [10, Theorem 2], we show that a proper choice of  $\gamma_n$  will ensure that the polynomial  $P_n(\mathbf{x}) + \gamma_n |\mathbf{x}|^{2n}$  has the desired properties for large  $n$ . Namely, we choose  $\gamma_n$  so that: (i) the weighted norm of the added term  $\gamma_n |\mathbf{x}|^{2n}$  is small, and, (ii) the second directional derivatives of the term are larger than those of  $P_n$  to ensure that the sum is convex on  $\mathbb{R}^d$ . In order to establish property (ii), we estimate the second directional derivatives of  $P_n$  using Lemma 6. This is the most technical part of the proof which leads to an additional constraint on  $r_n$ , namely that  $r_n$  grows faster than  $n^{1/\alpha}$ . Ultimately, the required choice of  $\gamma_n$  is possible if one selects  $r_n = n^{1/\beta}$  with some  $\beta$ ,  $1 < \beta < \alpha$ .

Now we show all the technical details in a formal proof.

*Proof of Theorem 1.* Suppose that  $\alpha > 1$ ,  $1 \leq p \leq \infty$ ,  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is convex, and  $fW_\alpha \in L_p(\mathbb{R}^d)$  if  $1 \leq p < \infty$ , or  $f \in C_{W_\alpha}(\mathbb{R}^d)$  if  $p = \infty$ . Let  $\varepsilon > 0$  be fixed.

We apply Lemma 4 to obtain a piecewise linear convex function  $h$  satisfying (3), and by (4),

$$(9) \quad \omega(h, t, \mathbb{R}^d) \leq Lt, \quad t > 0, \quad \text{and} \quad |h(\mathbf{x})| \leq L|\mathbf{x}| + |h(\mathbf{0})|, \quad \mathbf{x} \in \mathbb{R}^d,$$

where  $L > 0$  does not depend on  $t$ .

Fix some  $\beta$ ,  $1 < \beta < \alpha$ , and set  $r_n := n^{1/\beta}$ . We apply Lemma 3 with  $r = r_n$  and get a sequence of polynomials  $P_n \in \Pi_{n,d}$  convex on  $B(r_n)$  such that

$$(10) \quad \|h - P_n\|_{L_\infty(B(r_n))} \leq c_2 \omega \left( h, \frac{r_n}{n+1}, B(r_n) \right).$$

Taking into account (9), we have

$$(11) \quad \|h - P_n\|_{L_\infty(B(r_n))} \leq c_2 L \frac{r_n}{n+1} \leq c_2 L n^{1/\beta-1}.$$

In addition, since  $\alpha > \beta$  and both  $n$ th Freud and Mhaskar-Rakhmanov-Saff numbers have the order of  $n^{1/\alpha}$ , there exists  $n_0$  such that

$$r_n \geq \max\{4q_{2n}, 2a_n\}, \quad n \geq n_0.$$

Then Lemma 5 implies that for  $n \geq n_0$

$$\|P_n W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r_n))} \leq \begin{cases} 2^{-n} \|P_n W_\alpha\|_{L_\infty(B(q_{2n}))}, & \text{if } p = \infty, \\ c_3 e^{-c_4 n^\delta} \|P_n W_\alpha\|_{L_p(B(a_n))}, & \text{if } 1 \leq p < \infty. \end{cases}$$

The above inequality immediately implies

$$(12) \quad \|P_n W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r_n))} \leq e^{-c_5 n^{\delta'}} \|P_n W_\alpha\|_{L_p(B(r_n))}, \quad n \geq n_0,$$

with positive constants  $c_5$  and  $\delta'$  independent of  $n$ .

It is not hard to see that the polynomial  $P_n$  approximates  $h$  “well” in the weighted norm. Indeed, from (11), (12) and the fact that  $\|h W_\alpha\|_{L_p(\mathbb{R}^d)} < \infty$ , we conclude that there exists  $n_1 \geq n_0$  such that for all  $n \geq n_1$ ,

$$\|(h - P_n) W_\alpha\|_{L_p(B(r_n))} \leq \|h - P_n\|_{L_\infty(B(r_n))} \|W_\alpha\|_{L_p(\mathbb{R}^d)} < \varepsilon,$$

and

$$\begin{aligned} \|(h - P_n) W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r_n))} &\leq \|h W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r_n))} + \|P_n W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r_n))} \\ &\leq \|h W_\alpha\|_{L_p(\mathbb{R}^d \setminus B(r_n))} + e^{-c_5 n^{\delta'}} (\|(h - P_n) W_\alpha\|_{L_p(B(r_n))} + \|h W_\alpha\|_{L_p(B(r_n))}) \\ &< \varepsilon. \end{aligned}$$

Thus,

$$(13) \quad \|(h - P_n) W_\alpha\|_{L_p(\mathbb{R}^d)} < 2^{1/p} \varepsilon, \quad n \geq n_1.$$

Finally, consider new polynomials

$$S_n(\mathbf{x}) := P_n(\mathbf{x}) + Q_n(\mathbf{x}) := P_n(\mathbf{x}) + \varepsilon \tau_n |\mathbf{x}|^{2n}, \quad \mathbf{x} \in \mathbb{R}^d,$$



where  $\tau_n := \|\cdot\|_{L_p(\mathbb{R}^d)}^{-1} \| |^{2n} W_\alpha(\cdot) \|_{L_p(\mathbb{R}^d)}^{-1}$ . It is straightforward to verify that

$$(14) \quad \tau_n = \begin{cases} e^{2n/\alpha} \left( \frac{2n}{\alpha} \right)^{-2n/\alpha}, & \text{if } p = \infty, \\ \left( \frac{\alpha p^{(2np+d)/\alpha}}{dV_d \Gamma((2np+d)/\alpha)} \right)^{1/p}, & \text{if } 1 \leq p < \infty, \end{cases}$$

where  $V_d$  is the volume of the unit ball in  $\mathbb{R}^d$ . For  $p = \infty$ , evaluation of  $\tau_n$  is a simple optimization problem, and for  $1 \leq p < \infty$ , the  $L_p$ -norm that appears in the definition of  $\tau_n$  can be easily computed using spherical coordinates.

With this choice of  $\tau_n$ , by (3) and (13), we immediately get

$$(15) \quad \|(f - S_n)W_\alpha\|_{L_p(\mathbb{R}^d)} < (2^{1/p} + 2)\varepsilon, \quad n \geq n_1.$$

It remains to show that the choice of  $\tau_n$  also guarantees convexity of  $S_n$  on  $\mathbb{R}^d$  for sufficiently large  $n$ . Indeed, it is clear that  $Q_n$  is convex on  $\mathbb{R}^d$ , hence,  $S_n$  is convex on  $B(r_n)$ . From [10, (26)], we get that

$$(16) \quad \frac{\partial^2 Q_n(\mathbf{x})}{\partial \boldsymbol{\mu}^2} \geq 2n\varepsilon\tau_n |\mathbf{x}|^{2(n-1)}, \quad \mathbf{x}, \boldsymbol{\mu} \in \mathbb{R}^d, \quad |\boldsymbol{\mu}| = 1.$$

We need to verify that given any direction  $\boldsymbol{\mu} \in \mathbb{R}^d$ ,  $|\boldsymbol{\mu}| = 1$ , we have

$$\frac{\partial^2 P_n(\mathbf{x})}{\partial \boldsymbol{\mu}^2} + \frac{\partial^2 Q_n(\mathbf{x})}{\partial \boldsymbol{\mu}^2} \geq 0, \quad |\mathbf{x}| \geq r_n.$$

Taking into account (7) from Lemma 6 and the inequality (16), it is sufficient to show that

$$2n\varepsilon\tau_n |\mathbf{x}|^{2(n-1)} \geq \left( \frac{8|\mathbf{x}|}{r_n} \right)^n \frac{8n(n-1)}{r_n^2} \|P_n\|_{L_\infty(B(r_n))}, \quad |\mathbf{x}| \geq r_n.$$

The inequalities (9) and (11) imply that

$$\|P_n\|_{L_\infty(B(r_n))} \leq \|h - P_n\|_{L_\infty(B(r_n))} + \|h\|_{L_\infty(B(r_n))} \leq c_2 L \frac{r_n}{n} + Lr_n + |h(\mathbf{0})| \leq c_6 r_n,$$

where  $c_6 > 0$  is independent of  $n$ . Therefore, we need to prove that for sufficiently large  $n$ ,

$$2n\varepsilon\tau_n |\mathbf{x}|^{2(n-1)} \geq c_7 \left( \frac{8|\mathbf{x}|}{r_n} \right)^n \frac{n^2}{r_n}, \quad |\mathbf{x}| \geq r_n,$$

where  $c_7 = 8c_6$ . Since  $2(n-1) \geq n$  for  $n \geq 2$ , it is enough to verify that the previous inequality holds when  $|\mathbf{x}| = r_n$ :

$$2n\varepsilon\tau_n (r_n)^{2(n-1)} \geq c_7 \frac{8^n n^2}{r_n},$$

which, recalling that  $r_n = n^{1/\beta}$ , is equivalent to

$$(17) \quad \ln(\tau_n) + \frac{2n-1}{\beta} \ln n \geq \ln c_7 + n \ln 8 + \ln n - \ln(2\varepsilon) =: I(n).$$

Clearly,  $\lim_{n \rightarrow \infty} \frac{I(n)}{n \ln n} = 0$ . If  $p = \infty$ , it readily follows from (14) that  $\lim_{n \rightarrow \infty} \frac{\ln(\tau_n)}{n \ln n} = -\frac{2}{\alpha}$ . If  $1 \leq p < \infty$ , we note that by the Stirling's formula  $\lim_{t \rightarrow \infty} \frac{\ln(\Gamma(t))}{t \ln t} = 1$ , therefore, by (14), we have  $\lim_{n \rightarrow \infty} \frac{\ln(\tau_n)}{n \ln n} = -\frac{2}{\alpha}$  in this case as well. Since

$$\lim_{n \rightarrow \infty} \frac{1}{n \ln n} \left( \ln(\tau_n) + \frac{2n-1}{\beta} \ln n - I(n) \right) = -\frac{2}{\alpha} + \frac{2}{\beta} > 0,$$

there exists  $n_2 \geq n_1$  such that for any  $n \geq n_2$  the inequality (17) holds.

This means that for any  $n \geq n_2$  the polynomial  $S_n$  of total degree  $2n$  is convex on  $\mathbb{R}^d$  and satisfies (15). Since  $\varepsilon > 0$  was arbitrary, the proof of the theorem is complete.  $\square$

#### ACKNOWLEDGEMENT

We thank both referees for their valuable comments one of which pointed to a serious typo in the initial submission of the manuscript.

#### REFERENCES

- [1] Ronald A. DeVore and George G. Lorentz, *Constructive approximation*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 303, Springer-Verlag, Berlin, 1993.
- [2] Michael Ganzburg, *Restricted range inequalities in multivariate weighted approximation*, Approximation Theory XI: Gatlinburg (2004), 175–184.
- [3] K. A. Kopotun, D. Leviatan, A. Prymak, and I. A. Shevchuk, *Uniform and pointwise shape preserving approximation by algebraic polynomials*, Surv. Approx. Theory **6** (2011), 24–74.
- [4] András Kroó, *On weighted polynomial approximation on the plane*, East J. Approx. **1** (1995), no. 1, 73–81.
- [5] András Kroó and Szilárd Révész, *On Bernstein and Markov-type inequalities for multivariate polynomials on convex bodies*, J. Approx. Theory **99** (1999), no. 1, 134–152.
- [6] Dany Leviatan and Doron S. Lubinsky, *The degree of shape preserving weighted polynomial approximation*, J. Approx. Theory **164** (2012), no. 2, 218–228.
- [7] D. S. Lubinsky, *A survey of weighted polynomial approximation with exponential weights*, Surv. Approx. Theory **3** (2007), 1–105.
- [8] Oleksandr Maizlish, *Shape preserving approximation on the real line with exponential weights*, J. Approx. Theory **157** (2009), no. 2, 127–133.
- [9] Yannis Sarantopoulos, *Bounds on the derivatives of polynomials on Banach spaces*, Math. Proc. Cambridge Philos. Soc. **110** (1991), no. 2, 307–312.
- [10] A. S. Shvedov, *Comonotone approximation of functions of several variables by polynomials*, Math. Sb. **115** (1981), no. 4, 577–589.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MB, R3T2N2, CANADA  
*E-mail address:* alexmaizlish@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MB, R3T2N2, CANADA  
*E-mail address:* prymak@gmail.com